Easy as Pi, you say?
Irrationality of π, Etc.

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Ray Redheffer was a Professor of Mathematics at UCLA, 1950–1991.

He enjoyed working with students.

By very good luck, in high school I was one of them.

He showed me a simple calculus proof that π is irrational.

Ray was physically fit into his eighties.

Ray died of cancer in 2005 at age 84.

He was a fine mathematician (over 200 publications) and much else besides.

- He encouraged students of all ages and gave popular lectures at public schools,
- created many of the exhibits in the famous Eames Mathematica Exhibit,
- built his own home,
- could do a one-arm chin-up,
- played classical music on his grand piano,
- and loved to recite long poems from memory.

Our subject today is π.

3.
1415926535 8979323846 2643383279 5028841971
6939937510 5820974944 5923078164 0628620899
8628034825 3421170679 8214808651 3282306647
0938446095 5058223172 5359408128 4811174502
8410270193 8521105559 6446229489 5493038196
4428810975 6659334461 2847564823 3786783165
2712019091 4564856692 3460348610 4543266482
1339360726 0249141273 7245870066 0631558817
4881520920 9628292540 9171536436 7892590360
0113305305 4882046652 1384146951 9415116094
3305727036 5759591953 0921861173 8193261179
3105118548 0744623799 6274956735 1885752724
. . . . . .

The decimal representation is non-repeating.

π *is* not *a rational number.*

- π is an **irrational** number;

  there are **no two integers** a and b such that $\pi = a/b$.

  That is what we will prove today.

- Even more is true: π is *transcendental*;

  there can be no integers $a_0, a_1, \ldots, a_n$ such that

  $a_0\pi^n + a_1\pi^{n-1} + \cdots + a_{n-1}\pi + a_n = 0$

  (Lindemann's theorem—it's too hard for today!)

I'll show you the simple calculus proof Ray showed me.

Ray said, "**Think about this polynomial and integral**:"

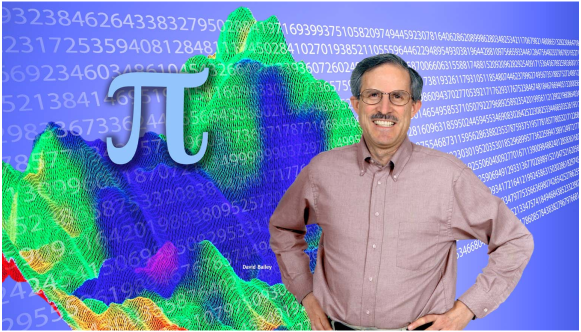
$$p_n(x) = \frac{x^n(a - bx)^n}{n!}, \quad I(n) = \int_0^{\pi} p_n(x) \sin x dx$$

Today I'll sketch the proof.

You can fill in the details later and show your friends!

First, some fun facts....

Two leading Pi-oneers, 1



(Photo provided courtesy of David H. Bailey)

David Bailey's Pi Directory:

http://crd.lbl.gov/~dhbailey/pi/

Two leading πioneers, 2

(Photo provided courtesy of Jon Borwein)

Jonathan Borwein's Pi Site:

www.cecm.sfu.ca/~jborwein/pi_cover.html

Their webs and books show many interesting things about π and more.

- $$\pi = \sqrt{12}\left(1 - \frac{1}{3\cdot 3} + \frac{1}{5\cdot 3^2} - \frac{1}{7\cdot 3^3} + \cdots\right)$$ (Madhava)

  Madhava invented analysis and computed first 13 digits of π **three centuries before Newton**! (Wikipedia)

- $$\pi = \frac{3\sqrt{3}}{4} + 24\left(\frac{1}{3\cdot 2^3} - \frac{1}{5\cdot 2^5} - \frac{1}{7\cdot 2^7} - \cdots\right)$$ (Newton)

  Newton computed the first 16 digits of π.

- $$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801}\sum_{k=0}^{\infty}\frac{(4k)!(1103+26390k)}{(k!)^4 396^{4k}}$$ (Ramanujan)

Here are two more amazing things.

The following formula was **discovered with the help of computer programs**. It can be used on PC's to find **any** single binary or hexadecimal digit of π (to the qaudrillionth and beyond!):

- $\pi = \sum_{i=0}^{\infty} \frac{1}{16^i} \left(\frac{4}{8i+1} - \frac{1}{8i+4} - \frac{1}{8i+5} - \frac{1}{8i+6} - \cdots \right)$

(D. H. Bailey, P. Borwein, S. Plouffe, 1996)

- Japanese scientists have computed π to over a **trillion** decimal places.

(Kanada, et al, 2002)

Let's do the Math: π is irrational! *(Sketch of proof-by-contradiction)*

Suppose π **is** rational, say, $\pi = a/b$; **derive a contradiction**:

Let

$$p_n(x) \equiv \frac{x^n(a-bx)^n}{n!}, \quad \text{so,} \quad p_n(\pi - x) = p_n(x)$$

Denote $p_n^k(x) \equiv \frac{d^k}{dx^k} p_n(x)$, and notice,

$$p_n^k(\pi - x) = \pm p_n^k(x), \quad k \in \{0, 1, 2, \ldots, 2n\}$$

Therefore, $p_n^k(0)$ **is an integer, and therefore so is** $p_n^k(\pi)$**.**

Here are some hints for you to continue the proof.

For a contradiction, show that the integral $I(n)$ has a **split personality**: **(it's an integer, but it isn't)**,

$$I(n) = \int_0^{\pi} p_n(x) \sin x dx$$

Because

1. $p_n(x) > 0$ for $x \in (0, \pi)$, and for large enough n,

   $p_n(x)$ is **arbitrarily small.**

2. **So**, for very large n, $0 < I(n) < 1$, **therefore**

   $I(n)$ **is not** an integer!

3. **But**, integrate $I(n)$ by parts $2n$ times to show that

   $I(n)$ **is** an integer!

OK, here's one extra hint!

To prove Item 3 on the last slide, **show that**:

- $p_n(x)$ is a polynomial with **all integer coefficients** **times** a factor of $x^n/n!$ at each term;
- therefore, **the constant term** of $p_n^k(x)$ is an integer (can be 0);
- **therefore**, $p_n^k(0)$ **is an integer**, so $p_n^k(\pi)$ **is, too**!

Apply the last "bullet" after you integrate $I(n)$ by parts $2n$ times.

References

- L. Berggren, J. Borwein & P. Borwein, *Pi: A Source Book*, Springer, 2004
- J. Borwein and D. Bailey, *Mathematics by Experiment*, A.K. Peters, 2004
- T. W. Gamelin, "IN MEMORIAM, Raymond Redheffer, Professor of Mathematics, Emeritus, Los Angeles, 1921-2005." (Accessible on Internet)
- M. Raugh, "Ray Redheffer remembered 2005." (MR's web)

For more about π see webs of Πioneers David Bailey and Jon Borwein referenced in previous slides.

I challenge you to fill in the details of the proof that π is irrational!

The slides for this talk will be available on my web:

http://mikeraugh.org

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