CIRCLING AN ELLIPSE

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Abstract. Here's a novel way to construct a circle. Take a carpenter's square and press it against an ellipse, making two points of contact by aligning the inner edges of the "square" with the outer edge of the ellipse. The square is now tangent to the ellipse in two places. While keeping the square snug against the ellipse, revolve the square around the ellipse. So what shape does the inner vertex of the carpenter's square trace out in its revolution? A circle!

1. Introduction

What a peculiar construction, pushing a carpenter's square around an ellipse! That this will produce a circle may seem surprising to you, as it did to me. The carpenter's square has to be large enough to embrace the ellipse in all positions.

The construction illustrates a property of ellipses that strikes me as fundamental, causing me to think there must be some basic theory of conics to account for it. I say a little more about this at the end.

The problem is like a lock. The keyhole is in plain view, but the tumblers have to be aligned carefully. The keyhole is this: If the ellipse can be circled in the way described, then the circle must be co-centered with the ellipse, and from any point on the circle the two tangent lines from the point to the ellipse must be orthogonal. Now let us align the tumblers!

2. Setting it up

Specify an ellipse with typical point (ξ, η) centered at the origin, with major semi-axis a and minor semi-axis b:

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Acknowledgement. The mathematician H. G. Senge posed this problem to me in the early 1960's. After seeing this article he told me he can't recall where he had first seen the problem but after recent searching found that the circle of this article has been known as the *director circle* for the underlying ellipse. Searching the Internet, I have found references to the director circle in projective geometry going back to the 1800s. I plan to write more about this in a revision.

$$\frac{\xi^2}{a^2} + \frac{\eta^2}{b^2} = 1 \tag{1}$$

Tumbler 1. Let's *assume* the vertex of the carpenter's square sweeps out a circle centered at the origin, an assumption we are going to prove. As a warm-up exercise, you can deduce that if the locus of the vertex really is a circle, then the radius must be $R = \sqrt{a^2 + b^2}$, a fact we'll use and in so doing verify. But check it out!

Let's be very clear about the nature of the proof we are following. We have anticipated the solution to the problem, namely that from any point (x, y) on the circle of radius $R = \sqrt{a^2 + b^2}$ co-centered with the ellipse, the two rays emanating from (x, y) that are tangent to the ellipse will be orthogonal. This orthogonality is what we must prove.

Pick a fixed point in the plane of the ellipse, say (x, y) where $x^2+y^2 = R^2$, and characterize a tangent line emanating from (x, y) to the ellipse. Now constrain the point (ξ, η) on the ellipse to be such a point of tangency, so that,

$$\frac{y - \eta}{x - \xi} = -\frac{b^2}{a^2}\frac{\xi}{\eta} \tag{2}$$

Or equivalently,

$$\frac{x\xi}{a^2} + \frac{y\eta}{b^2} = \frac{\xi^2}{a^2} + \frac{\eta^2}{b^2} = 1 \tag{3}$$

Tumbler 2. Since the coordinates (x, y) and the semi-axes a and b are fixed values, Equations (1) and (3) can be reduced to a single quadratic equation to *determine* either ξ or η, whichever we choose. The quadratic will produce the two possible points of tangency.

Tumbler 3. To obtain such a quadratic, we can make an arbitrary choice to eliminate either η or ξ from the two equations. But we shall see that it is useful to obtain *both* quadratics.

For example, to find the quadratic for ξ we begin by eliminating η. From Eq. (3),

$$\eta = \frac{b^2}{y}\left(1 - \frac{x\xi}{a^2}\right)$$

Substitute this value for η into Eq. (1) to get the quadratic equation for ξ,

$$(a^2y^2 + b^2x^2)\xi^2 - 2a^2b^2x\xi + a^4\left(b^2 - y^2\right) = 0 \tag{4}$$

We can use symmetry to find the quadratic for η. Eqs. (1) and (3) are invariant on replacing x, ξ, a with y, η, b. So the quadratic equation for η can be obtained by interchanging those variables in the previous quadratic:

$$(a^2y^2 + b^2x^2)\eta^2 - 2a^2b^2y\eta + b^4\left(a^2 - x^2\right) = 0 \tag{5}$$

3. Deductions

Keep it in mind that we have anticipated the solution to the problem and are in the process of proving our guess was right. We have chosen a point (x, y) on the circle of radius $R = \sqrt{a^2 + b^2}$ and have derived equations for $(\xi_i, \eta_i), i = 1, 2$, the two points of tangency determined by lines from (x, y). All we have to do now is prove that the two tangent lines emanating from (x, y) are orthogonal.

Tumbler 4. A quadratic has two solutions, so the two solution must yield the two tangent segments to the ellipse from the point (x, y). We now prove that the two segments are orthogonal by showing that their *dot product* is equal to zero.

Assuming the two solutions (ξ_1, η_1) and (ξ_2, η_2) thus derived from either Eqs. (4) and (3)—or from Eqs. (5) and (3)—we want to demonstrate orthogonality of the two segments $(x - \xi_1, y - \eta_1)$ and $(x - \xi_2, y - \eta_2)$. In other words we want to show that,

$$\begin{aligned} (x - \xi_1, y - \eta_1) \cdot (x - \xi_2, y - \eta_2) &= \\ x^2 + y^2 - (\xi_1 + \xi_2)x - (\eta_1 + \eta_2)y + (\xi_1\xi_2 + \eta_1\eta_2) &= 0 \end{aligned} \tag{6}$$

Tumbler 5. The evaluation of the expression at Eq. (6) is simplified by noting that the parenthetical expressions are symmetric functions of the roots of quadratic Eqs. (4) and (5)—they can be evaluated *directly* from the coefficients of those equations. This observation eliminates the necessity of actually solving either of the quadratic equations, sparing some algebra. The two quadratics yield with ease the following crucial information:[1]

$$
\begin{aligned}
\xi_1 + \xi_2 &= \frac{2a^2b^2x}{a^2y^2 + b^2x^2} \\
\eta_1 + \eta_2 &= \frac{2a^2b^2y}{a^2y^2 + b^2x^2} \\
\xi_1\xi_2 + \eta_1\eta_2 &= \frac{a^4b^2 - a^4y^2}{a^2y^2 + b^2x^2} + \frac{b^4a^2 - b^4x^2}{a^2y^2 + b^2x^2} \\
&= \frac{a^4b^2 + b^4a^2}{a^2y^2 + b^2x^2} - \frac{a^4y^2 + b^4x^2}{a^2y^2 + b^2x^2}
\end{aligned}
$$

Open the Lock. Substituting the foregoing values for the symmetric functions in parentheses in Eq. (6)—and using the stated value for $x^2 + y^2$, namely, $R^2 = a^2 + b^2$—find that the expression on the left-hand side of (6) does in fact reduce to zero. The dot product vanishes, proving orthogonality of the two tangent segments.

Therefore, revolving a carpenter's square around an ellipse produces a circle.

4. A Musing Conclusion

So, what does it all mean? Since the same circle of radius R will be produced for all ellipses for which the semi-axes a and b satisfy $R = \sqrt{a^2 + b^2}$, the construction picks out a *family* of ellipses. This suggests that there may be something special about this family of conics—and that there may be a more "natural" way to characterize the same family, leading to a better proof.

You can consider the construction in a dual sense. While keeping the carpenter's square in a fixed position, rotate the ellipse within its embrace. Then the center of the ellipse will not remain fixed but will oscillate at a fixed distance R from the inner vertex of the square—it will oscillate on the arc of a circle. The exception is the degenerate case in which the ellipse is a circle. You might not have guessed this result if you didn't already know that the carpenter's square will trace a circle around the ellipse.

Here's something curious about the proof. If you make the substitution $b = ib$, so that b^2 is replaced by $-b^2$, all the algebra remains valid.

[1] Use these facts, $(\xi - \xi_1)(\xi - \xi_2) = \xi^2 - (\xi_1 + \xi_2)z + \xi_1\xi_2$, and $(\eta - \eta_1)(\eta - \eta_2) = \eta^2 - (\eta_1 + \eta_2)z + \eta_1\eta_2$.

This shows that you can use a carpenter's square to produce arcs of a circle from the branches of the hyperbla $\dfrac{\xi^2}{a^2} - \dfrac{\eta^2}{b^2} = 1$. But notice that in this case the square must have infinitely long sides.

Finally, a question you might want to consider: Can you construct a sphere by pushing a *corner cube* (think of it as a corner section of a cubical box) around an arbitrary ellipsoid?

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