

# KEPLER'S ELLIPTICAL ORBITS OF THE PLANETS AND NEWTON'S INVERSE-SQUARE LAW OF GRAVITATION

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ABSTRACT. Assuming Kepler's law that the planets travel in ellipses with the Sun at a focus, Newton answered these question: What acceleration is experienced by a point-mass moving on an ellipse subject only to a central force acting from the focus of the ellipse? How must the force change as a function of position on the ellipse? These questions lead to the inverse-square law of gravitation, as I show in this article. By starting with an ellipse, Newton avoided the more difficult reverse problem of starting with the inverse-square law and solving the differential equations of motion to infer an ellipse, a feat performed first by Johann Bernoulli in Newton's lifetime. Newton derived his results in the *Principia* using innovative geometric arguments suited for his audience, but I use what is more familiar today—vector notation and differential calculus—to derive Newton's results and, to demonstrate the contrast in difficulty, solve the differential equations of motion in the style of Bernoulli.

## 1. INTRODUCTION

The background of Newton's discovery of universal gravitation is an interesting topic in itself. I touch on just a few facets and provide references for anyone who would like to go deeper.

Newton was not the first on record to state that elliptic planetary motion could be the result of the interaction of the inertia of a planet and a central attractive force inversely proportional to the square of the distance of the planet from the center. That honor may be due to Robert Hooke documented in an exchange of letters with Newton near the end of 1679 in which Newton expressed the contrasting idea that if a body could fall unobstructed within the Earth toward its center it would wind in a spiral (Iliffe [5, Pages 83-86]). It is not clear what assumptions Newton was making, but if he thought that the center of the Earth would attract an object with the same force as it would an object falling from outside the surface, he was clearly in error in two ways: First, in that case the trajectory would be an

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ellipse as he later proved in the *Philosophiae Naturalis Principia Mathematica* (*Principia* below). Second, he also proved there that the central force within a homogeneous Earth would be proportional to the distance from the center; had he known this at the time of his letter, he would have recognized that the curvature of the trajectory in an inner-Earth regime would *decrease* as it approached the center—not increase as in a spiral. It appears that at this time, when Newton was already 36 years old, he was unaware of or had forgotten the inverse-square law and its implications.<sup>1</sup>

However, when Newton had begun preparing work on the *Principia* by April of 1685, he had already established the inverse-square law as the basis for his theory of universal gravitation.<sup>2</sup> He began by assuming Kepler's law that planets travel in ellipses with the Sun at a focus.<sup>3</sup> He generalized Kepler's area law to show that a point-mass accelerated by an arbitrary central force would move in a trajectory such that the radius vector would sweep out area at a constant rate. From this he deduced the inverse-square law for the central force required to maintain a planet on an elliptical trajectory. As we shall see, these results would not have been difficult for Newton to obtain, since he already had his version of *fluxional* calculus well under control.<sup>4</sup>

Newton did not formulate and solve the problem of planetary motion in terms of differential equations. It would have been feasible

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<sup>1</sup>Newton had already deduced the centripetal force required to keep an object in a circular orbit in his youthful *waste book* (Iliffe [5, Pages 26-27]. But this is a relatively simple problem because the radius of the trajectory is constant and its tangential speed can be specified as a constant. The correspondence implies that Newton had not yet taken the more difficult steps to determine the central force required to keep an object traveling in an elliptical orbit at an appropriately varying speed.

<sup>2</sup>[http://en.wikipedia.org/wiki/Isaac\\_Newton/Authoring\\_Principia](http://en.wikipedia.org/wiki/Isaac_Newton/Authoring_Principia), accessed March 7, 2007.

<sup>3</sup>"Despite [Newton's] later dismissal of Hooke's abilities and his refusal to continue the [exchange cited above] any further, Newton later confessed to Edmund Halley that this exchange incited him to think anew about celestial mechanics. Indeed, it was probably about this time that Newton momentarily used Kepler's Second Law to demonstrate that on an elliptical orbit a body is subject to an inverse-square law of attraction." (Iliffe [5, Page 86])

<sup>4</sup>As early as 1665 at age 22 Newton had begun developing his methods of calculus and had discovered the Fundamental Theorem.

for him to do so using his calculus methods, which he shared privately among a few close associates.<sup>5</sup> But he presented his results to the public in the *Principia* using an original method of geometric analysis that he stated would be more easily understood by his contemporaries familiar with classical geometry. The more difficult task of formulating and solving the differential equations of motion assuming a central inverse-square-law of attraction was accomplished first by Johann Bernoulli in Basel, based on the version of *differential and integral* calculus advanced by Leibniz. These methods were eventually crystallized by Euler, and with rigor contributed in later generations by Cauchy, Weierstrass, Riemann, Dedekind and Cantor, they formed the basis for modern calculus. (See Guicciardini [3] and Dunham [2].)

In the following demonstrations I begin on the same path of reasoning taken by Newton: I start by deriving Newton's area law for a body accelerating under the influence of a central force. Then, assuming motion on an ellipse, I deduce the inverse-square law without solving the differential equations of planetary motion. In the appendices I provide some alternative derivations for comparison. In the first appendix, I begin with the differential equations of motion, and assuming an inverse-square law of force, I solve the equations in much the same manner as Johann Bernoulli. In the second appendix, I show two ways that a point-mass accelerating due to a central force must move in a plane, using a simple geometric argument followed by a rigorous calculus argument.

I use differential calculus instead of geometry in the style of Newton, but, of course, I must also use Newton's laws of motion:

1) Every object in a state of uniform motion tends to remain in that state of motion unless an external force is applied to it. (This is Galileo's law of inertia.)

2) The change in motion is proportional to the motive force impressed and is made in the direction of the straight line in which that force is impressed ( $\mathbf{F} = m\mathbf{a}$ ).

3) To every action there is always an opposed and equal reaction.

Newton's laws are so familiar now that we take them for granted and often apply them implicitly. Even the notion of the *mass* of an object, and the parallelogram law for adding vectors, are due to Newton. But these were revolutionary when Newton put them forward,

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<sup>5</sup>Newton was able to formulate and solve dynamical problems using fluxional calculus [3].

and they stand with his calculus at the cornerstone of his theory of mechanics. Although Hooke may have been first to conceive of the inverse-square law, he could not have demonstrated its consistency with elliptical orbits without calculus-like methods and the laws of motion, and these he did not have.

## 2. NEWTON'S AREA LAW

We imagine the motion of an object of point-mass  $m$  expressed as a vector in polar coordinates,

$$\mathbf{R} = r\mathbf{r}$$

where  $\mathbf{r} = (\cos \theta, \sin \theta)$  is a unit radius vector, with  $\theta$  measured counterclockwise in radians.

Differentiating twice with respect to time, we obtain acceleration  $\ddot{\mathbf{R}}$ .<sup>6</sup> First, the velocity vector,

$$\dot{\mathbf{R}} = \dot{r}\mathbf{r} + r\dot{\theta}\mathbf{r}_\theta$$

where  $\mathbf{r}_\theta = (-\sin \theta, \cos \theta)$  is a unit vector orthogonal to  $\mathbf{r}$  and points counterclockwise. And, then, the acceleration vector,

$$\ddot{\mathbf{R}} = (\ddot{r} - r\dot{\theta}^2)\mathbf{r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\mathbf{r}_\theta$$

Assuming central attraction, it must be that the component of force in the direction of  $\mathbf{r}_\theta$  vanishes:

$$2\dot{r}\dot{\theta} + r\ddot{\theta} = \frac{d}{dt}(r^2\dot{\theta}) = 0$$

The latter equation may be written  $\dot{A} = 0$ , so

$$(1) \quad A(t) = r^2\dot{\theta} = \alpha$$

with  $\alpha$  constant, is (twice) the rate at which area is swept out by the radius vector. This is Newton's area law.<sup>7</sup>

Incidentally, assuming the central force is of the form  $\mathbf{F} = -f\mathbf{r}$ , the expression above for  $\ddot{\mathbf{R}}$  allows us to write the equations of motion as

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<sup>6</sup>The convention of using *dots* over a variable to indicate differentiation with respect to time is due to Newton.

<sup>7</sup>Newton proved more, namely that the trajectory of an object accelerated by a central force lies within a plane. I prove this in an appendix, using it as an instance for which a convincing geometric argument is much simpler than a rigorous calculus argument.

follows:

$$(2) \quad \begin{aligned} \ddot{r} - r\dot{\theta}^2 &= -\frac{f}{m} \\ 2\dot{r}\dot{\theta} + r\ddot{\theta} &= 0 \end{aligned}$$

Assuming  $f = Gm/r^2$ , these are in essence the equations Johann Bernoulli solved. And to exhibit the comparative difficulty, I solve them in an appendix.<sup>8</sup>

### 3. THE INVERSE-SQUARE LAW

We derive the central force required to hold planets in elliptical orbits with the Sun positioned at a focus. We assume the fact that an object accelerated by a central force travels in a planar trajectory, a fact proved in an appendix. But here it is good enough to assume with Kepler that planetary motion is planar and elliptic. The equation of an ellipse with focus at the origin and major axis lying on the  $X$ -axis is given by<sup>9</sup>

$$(3) \quad \mathbf{R}(\theta) = \frac{L}{1 + e \cos \theta} \mathbf{r}(\theta), \quad 0 < e < 1$$

Since acceleration of an object is related to the applied force, we must deal with  $\dot{\mathbf{R}}$  and  $\ddot{\mathbf{R}}$ .

$$(4) \quad \dot{\mathbf{R}} = L \frac{e \sin \theta \dot{\theta}}{(1 + e \cos \theta)^2} \mathbf{r}(\theta) + L \frac{\dot{\theta}}{1 + e \cos \theta} \mathbf{r}_\theta(\theta)$$

And

$$\begin{aligned} \ddot{\mathbf{R}} = L \left[ \frac{2e^2 \sin^2 \theta \dot{\theta}^2}{(1 + e \cos \theta)^3} + \frac{e \cos \theta \dot{\theta}^2}{(1 + e \cos \theta)^2} + \frac{e \sin \theta \ddot{\theta}}{(1 + e \cos \theta)^2} - \frac{\dot{\theta}^2}{1 + e \cos \theta} \right] \mathbf{r} \\ + L \left[ \frac{2e \sin \theta \dot{\theta}^2}{(1 + e \cos \theta)^2} + \frac{\ddot{\theta}}{1 + e \cos \theta} \right] \mathbf{r}_\theta \end{aligned}$$

We apply Newton's second law stating that force is equal to mass times acceleration. Since we assume that the force  $\mathbf{F}(\theta)$  on the particle is directed toward the origin, the fact that  $\mathbf{r}$  is perpendicular to  $\mathbf{r}_\theta$  implies that the  $\mathbf{r}_\theta$ -component of the force must vanish. Noting

<sup>8</sup>I use  $G$  to represent the strength of the central force per unit of mass of the object being accelerated.

<sup>9</sup>This is a typical parameterization of a conic section:  $L$  is the semi-latus rectum, and  $e$  characterizes the type of conic. If  $0 \leq e < 1$ , then the conic is an ellipse (and  $e$  is its *astronomical eccentricity*); if  $e = 1$  the conic is a parabola, and if  $e > 1$  then the conic is a hyperbola.

that a similar quantity appears in the  $r$ -component, we can remove it, too, and wind up with

$$\begin{aligned}\mathbf{F}(\theta) = m\ddot{\mathbf{R}} &= mL \left[ \frac{e \cos \theta \dot{\theta}^2}{(1 + e \cos \theta)^2} - \frac{\dot{\theta}^2}{1 + e \cos \theta} \right] \mathbf{r} = -\frac{mL\dot{\theta}^2}{(1 + e \cos \theta)^2} \mathbf{r} \\ &= -\frac{mr^2\dot{\theta}^2}{L} \mathbf{r}\end{aligned}$$

But we saw in Equation (1) of the preceding section that, for a central force,  $r^2\dot{\theta} = \alpha$ , which substituted in the preceding expression yields the inverse-square law:

$$(5) \quad \mathbf{F}(\theta) = -\frac{m\alpha^2}{Lr^2} \mathbf{r}$$

So here we have the inverse-square law. Supposing the orbit is a circle of radius  $L$  and the particle of mass  $m$  is moving at a constant tangential speed  $v$ , then by Equation (1)  $\alpha^2 = L^4\dot{\theta}^2 = L^2v^2$ . Substitution in Equation (5) yields the result recorded in Newton's *waste book* mentioned in a footnote above, obtained there by Newton using simpler reasoning:

$$\mathbf{F}(\theta) = -\frac{mv^2}{L} \mathbf{r}$$

#### 4. CONCLUSION

We have not yet solved the differential equations of planetary motion, nor did we show that elliptical motion is even possible when driven by a central force.<sup>10</sup> We did something simpler: assuming the truth of Kepler's assertion that planets in fact do move in elliptical orbits, we inferred that the central attraction accelerating the planets must be an inverse square.

The combination of practical knowledge about the actual working of a physical system, combined with a method of geometric analysis that allowed Newton to demonstrate his results with convincing diagrams,<sup>11</sup> was typical of Newton's way of working. His criticism of Leibniz's approach was that it relied too much on purely symbolic manipulation of differentials and abstract reasoning, leaving it open to unchecked errors. Newton's criticism was well founded in the sense that later generations of mathematicians did find a need for

<sup>10</sup>This is done in the first appendix.

<sup>11</sup>I give an example of this in the second appendix where I show that a central-force trajectory is planar.

greater rigor in the methods of analysis,<sup>12</sup> nevertheless the Leibniz school prevailed in the end. And, indeed, the mathematical techniques we used above are rooted as much in the school of Leibniz as in the school of Newton.

But Newton did more than simply derive the inverse-square law from the assumption of an elliptical orbit; he also showed that a point-mass with initial given velocity moving under the influence of a central inverse-square law would travel in a unique conic. This, however, is not the same as formulating the problem of planetary motion as differential equations and solving them systematically, deducing an ellipse from the assumption of the inverse-square law. That was accomplished by a later generation of mathematicians.

Much of Newton's thought seems foreign to us today. The great English economist John Maynard Keynes was an avid student of Newton's writings in science and theology as well as a collector of Newton's writings in alchemy. Here is what he had to say at the home of Newton on Christmas Day, 1942, on the three-hundredth anniversary of Newton's birth:

"[Newton] came to be thought of as the first and greatest of the modern age of scientist, a rationalist, one who taught us to think on the lines of cold and untinctured reason.... Newton was not the first of the age of reason. He was the last of the magicians, the last of the Babylonians and Sumerians, the last great mind which looked out on the visible and intellectual world with the same eyes as those who began to build our intellectual inheritance rather less than 10,000 years ago." [7, *Newton The Man*, Pp 277-285].

I have drawn on several sources. Additional resources can be found in the following. Westfall [8] has written a critically acclaimed nine-hundred-plus-page biography that sets Newton in the social and scientific context of his time. In fewer pages Oxford University Press, in its Very Short Introduction series, shows something of the variety and remoteness of Newton's ideas in alchemy and theology as well as in science [5]. Guicciardini [3] reviews early criticism of Newton's work by Continental mathematicians. Classic examples of Newton's and Leibniz' use of their distinctive methods of calculus are in Dunham [2]. The textbook by Hahn [4] discusses Newton's

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<sup>12</sup>Newton's methods, too, came under criticism, as in the famous challenge of George Berkeley, in his essay *The Analyst*: "And what are these fluxions? The veocities of evanescent increments? And what are these same evanescent increments? They are neither finite quantities nor quantities infinitely small, nor yet nothing. May we not call them the ghosts of departed quantities?" Quoted by Dunham [2, p71].

planetary theory and compares the calculus of Newton and Leibniz. A derivation of Kepler's laws using potential theory can be found in the classic text of Courant and John [1].

## 5. APPENDIX: DERIVATION OF ELLIPSES FROM DIFFERENTIAL EQUATIONS OF MOTION

To demonstrate the technical challenge faced by Johann Bernoulli, we solve the equations of motion for a particle accelerating due to a central force inversely proportional to the square of the distance from the source. The techniques are similar to those used by Bernoulli in his original solution [3, Pp 223-225]. Bernoulli formulated the problem explicitly in terms of differential equations, just as we do. And his criteria for a satisfactory solution are very much like our own:

- 1) apply a general analytical procedure without assuming the result sought;
- 2) find a general integral that reveals that all the possible solutions have been found.

Bernoulli, as a follower of Leibniz, utilized the symbolic manipulation of differentials (a questionable business in the eyes of Newton). It was not until the time of mathematicians like Cauchy, Riemann and Weirstrass—and later Dedekind and Cantor—that the fundamental operations of calculus were fully justified by rigorous analytical methods [2]. In the following derivation note especially how we exploit relationships among the differentials.

We begin with the equations of motion for an object accelerating under a central inverse-square law of force (See Equations (2)):

$$(6) \quad \begin{aligned} \ddot{r} - r\dot{\theta}^2 &= -\frac{G}{r^2} \\ 2\dot{r}\dot{\theta} + r\ddot{\theta} &= 0 \end{aligned}$$

A constant  $\alpha$  appeared in the derivation of the area law derived above from the second equation:  $A(t) = r^2\dot{\theta} = \alpha$ . Using  $\dot{\theta} = \alpha/r^2$  in the first equation yields,

$$\ddot{r} = \frac{\alpha^2}{r^3} - \frac{G}{r^2}$$

We will solve the system on the assumption that at a given point  $(r_0, \theta_0)$  the tangential speed is given as  $v_{t0}$ . Letting  $v = \frac{dr}{dt}$  and noting



that  $\ddot{r} = v \frac{dv}{dr}$ , the latter equation becomes

$$v dv = \left( \frac{\alpha^2}{r^3} - \frac{G}{r^2} \right) dr$$

which integrates to

$$\frac{v^2}{2} - \frac{v_0^2}{2} = -\frac{\alpha^2}{2r^2} + \frac{G}{r} - \left( -\frac{\alpha^2}{2r_0^2} + \frac{G}{r_0} \right)$$

Or

$$v^2 = -\frac{\alpha^2}{r^2} + \frac{2G}{r} + C$$

where

$$(7) \quad C = v_0^2 + \frac{\alpha^2}{r_0^2} - \frac{2G}{r_0}$$

where  $v_0$  is the initial rate of change of *length* of the radius vector—not  $v_{t_0}$  the tangential speed of the point-mass.

Now we can use the fact that

$$v = \frac{dr}{dt} = \frac{dr}{d\theta} \frac{d\theta}{dt} = \dot{\theta} \frac{dr}{d\theta} = \frac{G}{r^2} \frac{dr}{d\theta}$$

in the preceding equation to obtain

$$d\theta = \frac{\alpha}{r\sqrt{Cr^2 + 2Gr - \alpha^2}} dr$$

Consulting a table of integrals,<sup>13</sup> we find the latter integral to be of the type,

$$\int \frac{dx}{x\sqrt{X}} = \frac{-1}{\sqrt{-a}} \cos^{-1} \left( \frac{bx + 2a}{x\sqrt{b^2 - 4ac}} \right)$$

where

$$X = \sqrt{a + bx + cx^2}$$

And in our case,

$$a = -\alpha^2, \quad b = 2G, \quad c = C$$

Therefore,

$$\theta - \theta_0 = -\cos^{-1} \left( \frac{Gr - \alpha^2}{r\sqrt{G^2 + \alpha^2 C}} \right) + \cos^{-1} \left( \frac{Gr_0 - \alpha^2}{r_0\sqrt{G^2 + \alpha^2 C}} \right)$$

<sup>13</sup>The integral can be routinely computed beginning with the substitution  $x = 1/u$ . Jeffrey [6] gives the integral in terms of the *arc sine*, but the use of the negative *arc cosine* is equivalent. Newton as well as Leibniz and his followers knew the fundamental theorem of calculus, so by differentiation and assorted integration techniques including substitution of variables they built up catalogs of integrals, just as our present-day table-builders do.

And,

$$\frac{\alpha^2}{r} = G + \left( \sqrt{G^2 + \alpha^2 C} \right) \cos \left[ \theta - \theta_0 - \cos^{-1} \left( \frac{Gr_0 - \alpha^2}{r_0 \sqrt{G^2 + \alpha^2 C}} \right) \right]$$

Letting  $\phi_0$  be the *arc cosine* expression in the brackets, we may write,

$$r = \frac{\alpha^2/G}{1 + e \cos(\theta - \theta_0 - \phi_0)}$$

with  $e = \sqrt{1 + \alpha^2 C/G^2}$ . Thus the solution is a conic, as anticipated. But there is some additional information here.

Notice that any initial values for  $r_0, \theta_0, v_0$  that preserve  $e$  and  $\theta_0 + \phi_0$  will result in the same solution curve. The fixed value of  $\theta_0 + \phi_0$  can be an arbitrary constant, but in order for the resulting conic to be of the standard form of Equation (3) in the section on the Inverse-Square Law we must have that  $\theta_0 + \phi_0 = 0 \pmod{2\pi}$ . In this case the solution of the equations of motion is

$$(8) \quad r = \frac{\alpha^2/G}{1 + e \cos \theta}$$

Going back to the definition of  $\phi_0$ , we infer that the constraint  $\theta_0 + \phi_0 = 0 \pmod{2\pi}$  requires that

$$r_0 = \frac{\alpha^2/G}{1 + e \cos \theta_0}$$

which says nothing more than that the initial point  $r_0, \theta_0$  lies on the specified conic. However, as we shall see, the fact that the value of  $e$  is preserved says something more useful. Using Equation (4) for  $\dot{\mathbf{R}}$  in the section on the Inverse-Square Law we obtain an expression for  $C$  in terms of tangential speed  $v_{t0}$ :

$$v_{t0}^2 = v_0^2 + \frac{L^2 \alpha^2}{r_0^2}$$

In the present section  $L = \alpha^2/G$ . Also, from the section on the Inverse-Square Law, we can revisit Equation (5) for  $\mathbf{F}$  to find that the constant of force there, namely,  $L\alpha^2$ , must be the same as the constant  $G$  used in this section's formulation of the equations of motion. Combining the fact that  $L = \alpha^2/G$  and  $L\alpha^2 = G$ , we find that  $\alpha^4 = G^2$ , and therefore,

$$\frac{L^2 \alpha^2}{r_0^2} = \frac{(\alpha^4/G^2) \alpha^2}{r_0^2} = \frac{\alpha^2}{r_0^2}$$

This allows us to write

$$v_{t_0}^2 = v_0^2 + \frac{\alpha^2}{r_0^2}$$

and to rewrite  $C$  as,

$$C = v_{t_0}^2 - \frac{2G}{r_0}$$

So, finally,

$$e = \sqrt{1 + \frac{2\alpha^2}{mG^2} \left( \frac{mv_{t_0}^2}{2} - \frac{mG}{r_0} \right)}$$

Since the value of  $e$  is preserved at all points on the curve, the value of the last parenthesis must also be constant, expressing *conservation of energy* on the curve. This is the more useful result referred to above.

That all solutions of the equations of motion for an inverse-square law are conic sections was first claimed (and proved) by Newton. We have found by integration all the trajectories satisfied by the equations of motion, thus meeting Bernoulli's criteria.

Derivations similar to above can be found in many text books. For a development in terms of gravitational potential and a general law of energy conservation see Chapter 6 of the text by Courant and John [1].

## 6. APPENDIX: CENTRAL-FORCE MOTION IS PLANAR

We prove that central-force motion is planar in two ways: the first is an instance in which a geometric argument makes the result seem obvious, and the second gives a proof using differential calculus. The latter is perhaps more satisfying to the modern mind, but the force of an argument in the style of the former was generally satisfactory for Newton.

First, consider the plane determined by the radius vector and the velocity vector at any given instant; there is no component of force acting orthogonal to the plane—hence no force to deflect the object (and its velocity vector) off the plane. The competing forces of “inertia” (Newton's first law), which continues in the plane, and the central force, which acts on the object along the radius from the center lying in the plane (Newton's second law), have their resultant within the plane (Newton's parallelogram law). So the object must continue its motion in the plane. This is the crux of Newton's argument in the *Principia*, in which he used infinitesimally refined polygonal approximations to obtain properties of motion of an object reacting to a central force.

In our second line of reasoning we use differential calculus to obtain a proof that is more rigorous by today's standard but is no more convincing. We begin by representing an object's radial position from the center of force as a function of time  $\mathbf{R}(t)$ , then derive the velocity vector  $\dot{\mathbf{R}}(t)$  and the acceleration vector  $\ddot{\mathbf{R}}(t)$ . From the latter we determine the constraints on the coordinates imposed by the central force, then use them to show that the velocity vector must lie in a fixed plane.

$$(9) \quad \mathbf{R} = r (\cos \theta \sin \phi, \sin \theta \sin \phi, \cos \phi)$$

Using the subscript as an index, obtain an orthonormal basis by the differentiation suggested in the subscript,

$$(10) \quad \begin{aligned} \mathbf{r} &= (\cos \theta \sin \phi, \sin \theta \sin \phi, \cos \phi) \\ \mathbf{r}_\theta &= (-\sin \theta, \cos \theta, 0) \quad \text{normalized} \\ \mathbf{r}_\phi &= (\cos \theta \cos \phi, \sin \theta \cos \phi, -\sin \phi) \end{aligned}$$

And

$$(11) \quad \begin{aligned} \dot{\mathbf{r}} &= \sin \phi \dot{\theta} \mathbf{r}_\theta + \dot{\phi} \mathbf{r}_\phi \\ \dot{\mathbf{r}}_\theta &= -\sin \phi \dot{\theta} \mathbf{r} - \cos \phi \dot{\theta} \mathbf{r}_\phi \\ \dot{\mathbf{r}}_\phi &= \cos \phi \dot{\theta} \mathbf{r}_\theta - \dot{\phi} \mathbf{r} \end{aligned}$$

So

$$(12) \quad \dot{\mathbf{R}} = \dot{r} \mathbf{r} + r \dot{\mathbf{r}} = \dot{r} \mathbf{r} + r \sin \phi \dot{\theta} \mathbf{r}_\theta + r \dot{\phi} \mathbf{r}_\phi$$

And

$$(13) \quad \begin{aligned} \ddot{\mathbf{R}} &= \ddot{r} \mathbf{r} + \dot{r} \dot{\mathbf{r}} \\ &+ \left( \dot{r} \sin \phi \dot{\theta} + r \cos \phi \dot{\theta} \dot{\phi} + r \sin \phi \ddot{\theta} \right) \mathbf{r}_\theta + r \sin \phi \dot{\theta} \dot{\mathbf{r}}_\theta \\ &+ \left( \dot{r} \dot{\phi} + r \ddot{\phi} \mathbf{r}_\phi \right) + r \dot{\phi} \dot{\mathbf{r}}_\phi \end{aligned}$$

So, after expanding  $\dot{\mathbf{r}}$ ,  $\dot{\mathbf{r}}_\theta$  and  $\dot{\mathbf{r}}_\phi$  in terms of  $\mathbf{r}$ ,  $\mathbf{r}_\theta$  and  $\mathbf{r}_\phi$  and consolidating,

$$(14) \quad \begin{aligned} \ddot{\mathbf{R}} &= \left( \ddot{r} - r \sin^2 \phi \dot{\theta}^2 - r \dot{\phi}^2 \right) \mathbf{r} \\ &+ \left( 2\dot{r} \sin \phi \dot{\theta} + 2r \cos \phi \dot{\theta} \dot{\phi} + r \sin \phi \ddot{\theta} \right) \mathbf{r}_\theta \\ &+ \left( 2\dot{r} \dot{\phi} - r \sin \phi \cos \phi \dot{\theta}^2 + r \ddot{\phi} \right) \mathbf{r}_\phi \end{aligned}$$

The assumption of central force requires that the components of  $\mathbf{r}_\theta$  and  $\mathbf{r}_\phi$  vanish. Therefore,

$$(15) \quad \begin{aligned} 2\dot{r} \sin \phi \dot{\theta} + 2r \cos \phi \dot{\theta} \dot{\phi} + r \sin \phi \ddot{\theta} &= 0 \\ 2\dot{r} \dot{\phi} - r \sin \phi \cos \phi \dot{\theta}^2 + r \ddot{\phi} &= 0 \end{aligned}$$

Now we are ready to prove that central-attraction motion is planar. It is sufficient to show that the velocity vector from Equation (12),<sup>14</sup>

$$\dot{\mathbf{R}} = (\dot{r}, r \sin \phi \dot{\theta}, r \dot{\phi})$$

must at any given time lie in a fixed plane. The plane is determined as the one orthogonal to both the radius vector from Equation (9),  $\mathbf{R} = r \mathbf{r}$ , and  $\dot{\mathbf{R}}$ . So, define a normal vector

$$(16) \quad \mathbf{N} = (r \mathbf{r}) \times \dot{\mathbf{R}} = (0, -r^2 \dot{\phi}, r^2 \sin \phi \dot{\theta}) = -r^2 \dot{\phi} \mathbf{r}_\theta + r^2 \sin \phi \dot{\theta} \mathbf{r}_\phi$$

We want to show that  $\mathbf{N}(t)$  points in a constant direction. It is allowable for the magnitude of  $\mathbf{N}(t)$  to change with time as long as the direction remains constant. But, in fact, we will show that  $\mathbf{N}$  remains constant as a function of time by proving that  $\dot{\mathbf{N}} = \mathbf{0}$ .

$$(17) \quad \begin{aligned} \dot{\mathbf{N}} &= \frac{d}{dt} (-r^2 \dot{\phi} \mathbf{r}_\theta + r^2 \sin \phi \dot{\theta} \mathbf{r}_\phi) \\ &= - (2r\dot{r}\dot{\phi} + r^2\ddot{\phi}) \mathbf{r}_\theta - r^2 \dot{\phi} \dot{\mathbf{r}}_\theta \\ &\quad + (2r\dot{r}\sin \phi \dot{\theta} + r^2 \cos \phi \dot{\theta} \dot{\phi} + r^2 \sin \phi \ddot{\theta}) \mathbf{r}_\phi + r^2 \sin \phi \dot{\theta} \dot{\mathbf{r}}_\phi \end{aligned}$$

Incorporating the basis derivatives from Equations (11) and consolidating yields,

$$\begin{aligned} \dot{\mathbf{N}} &= r (-2\dot{r}\dot{\phi} - r\ddot{\phi} + r \sin \phi \cos \phi \dot{\theta}^2) \mathbf{r}_\theta \\ &\quad + r (2\dot{r}\sin \phi \dot{\theta} + 2r \cos \phi \dot{\theta} \dot{\phi} + r \sin \phi \ddot{\theta}) \mathbf{r}_\phi \end{aligned}$$

From Equations (15), we obtain  $\dot{\mathbf{N}} = \mathbf{0}$ , as asserted.

We have just shown that the motion is planar, but our analysis reveals more. Knowing now that  $\mathbf{N}$ , defined by Equation (16), is constant allows us to write,

$$(18) \quad \begin{aligned} r^2 \dot{\phi} &= \alpha \\ r^2 \sin \phi \dot{\theta} &= \beta \end{aligned}$$

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<sup>14</sup>Using as coordinates the projections of a vector on the orthonormal basis  $\mathbf{r}$ ,  $\mathbf{r}_\theta$ , and  $\mathbf{r}_\phi$  from Equations (10).

where  $\alpha$  and  $\beta$  are constants. This proves that the expressions  $r^2\dot{\phi}$  and  $r^2 \sin \phi \dot{\theta}$  have constant values on any solution curve for the equations of central-force motion. Such expressions are called “integrals” of the system. Another discovery is the reversed symmetry in the coefficients of  $\mathbf{R}_\theta$  and  $\mathbf{R}_\phi$  evident in Equations (14) and (18). In this reversal we can infer the orthogonal reaction of a gyroscope to twists of the axis of rotation.

Finally observe that, since the motion is planar, we can without loss of generality assume the solution curve lies either on a polar great circle by fixing  $\theta$  at a constant value or on the equator by fixing  $\phi = \pi/2$ . You can check Equation (14) to see that in either case the system reduces to a system equivalent to Equations (2), and the respective Equation (18) yields Newton’s area law. Newton’s area law can also be derived directly from the fact that  $\mathbf{N}$  is constant; as the cross product of the radius vector and the velocity vector, the magnitude of  $\mathbf{N}$  gives the area of the parallelogram subtended by the radius vector and the velocity vector. I leave the latter inference as “an exercise for the reader.”

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